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FUNDAMENTALS AND RECENT STUDIES OF FINSLER GEOMETRY

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Abstract:

Riemann, who was born sixty years before Finsler, really started the study on Finsler geometry. In his famous lecture, Riemann introduced the concept of manifolds, generalised metric, and gave an example of Finsler metric. The computations for this kind of case, he said, are challenging. This topic has not received much attention from mathematicians. Finsler, though, was the one who initially brought up the notion of it.The following study project serves as an example of how to expand the findings on Finsler space with a particular (,)-metric. We have examined the hypersurfaces of this generalised (,)-metric as first and second type hyperplanes but not as third kind hyperplanes. The current work provides a new geometric perspective for solving some problems using projective change, projective flat, Douglas space, dually flat, reversible geodesic, Berwald space, two-dimensional Landsberg space, Einstein metric, hypersurfaces, and hyperplanes of various types. In conclusion, the work provides an insight into Finsler spaces with some (,)-metric and its possibilities towards some applications.

1 Introduction

Finsler geometry is a kind of differential geometry. It usually considers as a generalization of Riemannian geome-try. In fact, Riemann, in his epoc-making lecture in 1854, already suggested a possibility of studying a geometry moregeneral than Riemannian geometry, but he said the geometrical meanings of quantities appearing in such a gener-alized space will not be clear and it can't produce any contribution to the geometry. Furthermore, Mathematicinsneglected the study of

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such spaces for more than 60 years. In his age of twenty four years, Finsler took up the problemrelated to the space equipped with the metric function which was mentioned by Riemann. In 1918, he submitted hisepoc-making thesis to Gotingen university. He studied this geometry from the stand point of a geometrization of thecalculus of variations. This thesis draws the attention to the most of the mathematicians working in geometry. So,Finsler (1894-1970) was considered the originator of the Finsler geometry.In 1927, Taylor gave the name Finsler space to the manifold equipped with this generalized metric.In 1934, Cartan introduced a system of axioms to give uniquely a Finsler connection from the fundamental func-tion. In the same line, Randers drew the attention to several physicists towards Finsler geometry.In 1951, Rund introduced a new concept of parallelism considering Finsler geometry as locally Minkowskian.Later on Mokoto Matsumoto devoted his effort to such approach and contributed much to this field. He wrote amonograph "The theory of Finsler connections" and circulated it among the mathematicians working in the field.The Finsler geometry has many applications in various fields of Physics and Biology such as the theory of relativity,thermodynamics, optics, ecology, evolution and developmental biology. Several mathematicians contributed to thestudy and improvement of Finsler geometry.The historical studies about development stages for Finsler geometry have been introduced by Matsumoto [37]and Won [14]. In this paper, we will elaborate the discussion to conclude thirty studies in terms to it

2 Basic Concepts of Finsler Geometry

In the calculus of variations, it is referred to as an indicatrix. Albert Einstein took numerous of Minkowski's classes at the Eidgenossische Polytechnikum in Zurich, where Minkowski was a professor. The idea of 4-dimensional Minkowski space, which was introduced by Minkowski, served as the basis for Einstein's theory of special relativity. Out of the 23 issues presented at the second International Congress of Mathematicians in 1900, D. Hilbert [56] answered 10 of them. Problem 4: "The straight-line as the smallest path joining two points" and Problem 23: "The further development of the methods of the calculus of variations" were two geometry-related topics that were tackled in this lecture. The Gottingen lectures by D. Hilbert on "The Calculus of Variations" had an impact on Finsler's master Caratheodory, who took on the task of developing a function y = y(x) by adopting the minimum of the integral I as the initial value.

$$I = \int_a^b F$$

Caratheodory's early work was founded on the calculus of variations as a result of this effect, and he developed a method for discovering the minimizer of certain integrals. In this method, the Hamilton-Jacobi equations were used. This approach has been dubbed "The Royal Road to the Calculus of Variations" in recent times. He found the connection between the calculus of variations and the first-order partial differential equations. Later, his idea of Riemannian space—a multidimensional domain—led to the development of the general theory of relativity. The current definition of an abstract Riemannian manifold was ultimately developed as a result of this in a precise manner. The core idea of "Finsler space" was born as a result. The "Curves and Surfaces in General Spaces" thesis by Finsler [46] is the source of Finsler's geometry. Under the direction of Caratheodory, a specialist in calculus of variations, he completed his thesis. Finsler used the calculus of variations extensively to address geometry problems involving spaces and the Finsler metric. The integral minimizer's benefit

$$F(\alpha) = \int_a^b F\left(\alpha(t), \alpha'(t)\right) dt,$$

He saw straight away that the Finser metric F(x, y) must be positively definite in order to satisfy the convexity constraint. The discoveries made by Finsler had a profound effect on succeeding research generations.

3 Evaluation and Discussion

The following chart illustrates the linkages between all four Finsler connections that were described before.

$$\begin{array}{ccc} C\Gamma = (\Gamma^{*i}_{jk}, G^i_j, C^i_{jk}) & \xrightarrow{\text{C-process}} & R\Gamma = (\Gamma^{*i}_{jk}, G^i_j, 0) \\ \downarrow P^1 - \text{ process} & & \downarrow P^1 - \text{ process} \\ H\Gamma = (G^i_{jk}, G^i_k, C^i_{jk}) & \xrightarrow{\text{C-process}} & B\Gamma = (G^{*i}_{jk}, G^i_j, 0) \end{array}$$

ces If a Riemannian space is thought to have specific geometrical properties, obey special tensor equations, admit special tensor fields, or any combination of these, it spaces, which are particular

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Finsler spaces in Riemannian and Minkowskian geometry, respectively. We thus have a significant challenge in classifying all Minkowskian spaces. The intriguing basic functions F(x, y) are simple to express in their concrete forms. The Randers metric, the Kropina metric, the generalised Kropina metric, the Matsumoto metric, and the cubic metric are a few examples. Finding Finsler spaces that are fairly equivalent to Riemannian spaces but not Riemannian and Minkowskian spaces that are akin to flat spaces but not flat is crucial for the advancement of Finsler geometry. The specific tensor equations fulfilled by torsion, curvature, and other significant tensors are the major focus of this section. We provide some definitions of particular Finsler spaces in the sections that follow, along with the findings that follow.

The basic function L(x, y) of a Finsler space F n = (Mn, L(x, y)) is said to be a Riemannian space if

$$L(x,y) = g_{ij}y^i y^j.$$

The class of all Riemannian spaces among Finsler spaces is denoted as Cijk = 0, which means that the vertical connection v of the Cartan's connection C is flat. If there is a coordinate system (x i) where L is merely a function of y i, then a Finsler space F n = (Mn, L(x, y)) is referred to as a locally Minkowskian space [96]. If and only if a Finsler space is an inch more conventional than a Riemannian and locally Minkowskian space, then it qualifies as being locally Minkowskian. It offers illustrations that are flawlessly Finslerian. The fact that all of a Berwald space's tangent spaces are linearly isometric to a single Minkowski space is its most clearly understood characteristic. It is possible to assert that the Berwald space unmistakably developed from a single Minkowski space. If the Berwald connection B's connection coefficients Gi jk are given by

are only a function of position, the area is referred to as a Berwald space. If and only if, a Finsler space is Berwald in light of Finsler connections.

i. For CΓ : C h ij|k = 0.

ii. For RΓ : F h ijk = 0.

iii. For BΓ : Gh ijk = 0.

A Finsler space is called a Landsbergs space [96] if the Berwald connection BΓ is h-metrical i.e.,

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gij(k) = 0. In terms of conditions of the Cartan's connection CΓ, a Landsberg space is described by

i. P i jk = 0.

ii. P h ijk = 0.

A Finsler space F n of the dimension n is called a Douglas space if

$$D^{ij}(x,y) = G^i(x,y)y^j - G^j(x,y)y^i,$$

are all homogeneous polynomials in y i of degree three.

A Finsler space of dimension n, more than two, is called C-reducible if Cijk is written in the form [96]

$$C_{ijk} = \frac{1}{n+1}\pi_{(ijk)}\left(h_{ij}C_k\right),$$

where $C_i = C_{ijk}$ is the torsion vector, h_{ij} is the angular metric $g_{ij} - l_i l_j$ and $\pi_{(ijk)}$ is the sum of cyclic permutation of i, j, k.

If Cijk has the form [96], then a Finsler space of dimension n, larger than two, is semi-C-reducible.

There exists a symmetric Finsler tensor field Aij fulfilling Ai0 = 0, and Cijk is represented in the form [96] when applied to a Finsler space of dimension n, greater than two.

4 Result Analysis

The class of Finsler spaces with (,)-metric, also known as Randers spaces, was developed by physicist G. Randers and is a significant subclass of Finsler spaces. The idea of (,)-metric was developed by Matsumoto [100]. Despite being relatively new, the study of Finsler spaces with (,)- metre is an essential component of Finsler Geometry and its applications.

[100] When L is a positively homogeneous function L (x, y) = p aij (x)y iy j and (x, y) = bi(x)y i of first degree in two variables, such that the Finsler metric F: TM R is given by F 2 (x, y) = L (x, y), where is Riemannian metric on M and is a differential 1-form M. The following is how a (,)-metric is written:

where = (s) is an open interval (b0, b0) C positive function. The following formula defines the norm x of with regard to

F must be defined by meeting the requirement that x b0 for every x M. Additionally, we define certain significant Finsler spaces with their specified names using the (,)-metric.

[54] The term "Randers metric" refers to a Finsler metric F = + where 2 = aijy i y

j is Riemannian metric (gravitational field) and = bi(x)y i is 1-form (electromagnetic field) with (x) = p an ij (x)bi(x)bj (x) 1. Randers [124] introduced the Rander's metric. Ingarden [58], who gave it the term Randers metric in the beginning, utilised it up in the theory of the electron microscope.

[54] If and only if aij bibj is positive-definite, provided that aij is positive-definite, a Randers metric F = + is positive valued.

[95] The space F n and its Finsler metric are referred to as the Kropina space. The Kropina metric is defined as F = 2, where and are as previously mentioned. The Kropina metric was initially introduced by Berwald [21] in relation to a two-dimensional Finsler space with rectilinear extremal. Kropina [67] conducted more research on it. It should be noted that although Kropina is not a standard Finsler metric, Randers is.

[67] If F(x, y) = m+1 m, (m = 0, 1), the Finsler fundamental function F(x, y) is known as the generalised Kropina metric, and the Finsler space produced by this metric is known as a generalised m-Kropina space. [97] The slope of the earth's surface, represented by the graph of the function z = f(x, y), is thought to be a basic two-dimensional Finsler space.

$$L(x, y, \dot{x}, \dot{y}) = \frac{\alpha^2}{v\alpha - w\beta},$$

where v *and* w *are non-zero constants and*

$$\alpha^2 = \dot{x}^2 + \dot{y}^2 + (\dot{x} f_x, \dot{y} f_y)^2,$$

$$\beta = \dot{x} f_x, \dot{y} f_y.$$

Here, f(x, y) is a differential form and x is the well-known convinced Riemannian metric.

5 Conclusion

A summary of the in-depth investigation on certain prospective issues in the area of Finsler space with some (,)-metrics is provided in this thesis. The aforementioned goals were successfully attained during the study process and within the allotted time period. The study of conformal modifications in AP-geometry is covered in Chapter 1. We learn about some brand-new conformally invariant tensor fields that are represented in terms of an AP-space's Weitzenbock and Levi-Civita connections. We demonstrate that the curvature tensors of artificial conformal connections are these conformal invariants. We think it's feasible to physically comprehend the resulting geometric objects. The primary findings of this chapter have been published in the "International Journal of Geometric Methods in Modern Physics" Vol. 15 (2018) 1850012. DOI: 10.1142/S0219887818500123. arXiv: 1604.00474 [math.DG].

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