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ON FINSLER GEOMETRY AND APPLICATIONS IN MECHANICS: REVIEW AND NEW PERSPECTIVES

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Abstract:

Each point on a base manifold may, for instance, be supplied in Finsler geometry with coordinates defining its location and a set of one or more vectors expressing directions.There are many connections that are connected to different covariant derivatives including affine and nonlinear coefficients and the related metric tensor, which may typically rely on direction as well as location. Finsler geometry provides a lot of generality for explaining a variety of physical processes since it includes Riemannian, Euclidean, and Minkowskian geometries as special instances. Here, the main emphasis is on descriptions of finite deformation of continuous media. A review of earlier work involving the use of Finsler geometry in continuum mechanics of solids is conducted after consideration of the essential mathematical concepts and derivations. Then, by combining ideas from Finsler geometry with phase field theories from the area of materials science, a novel theoretical explanation of continua with microstructure is presented.

1 Introduction

Mechanical behavior of homogeneous isotropic elastic solids can be described by constitutive models that depend only on local deformation, for example, some metric or strain tensor that may generally vary with position in a body. Materials with microstructure require more elaborate constitutive models, for example, describing lattice orientation in anisotropic crystals, dislocation mechanisms in elastic-plastic crystals, or cracks or voids in damaged brittle or ductile solids. In conventional continuum mechanics approaches,

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such models typically assign one or more time- and positiondependent vector(s) or higher-order tensor(s), in addition to total deformation or strain, that describe physical mechanisms associated with evolving internal structure. Mathematically, in classical continuum physics [1–3], geometric field variables describing behavior of a simply connected region of a body depend fundamentally only on referential and spatial coordinate charts related by a diffeomorphism denoting corresponding points on the spatial and material manifolds covered by corresponding chart(s) and denotingtime. State variables entering response functions depend ultimately only on material points and relative changes in heir position (e.g., deformation gradients of first order and possibly higher orders for strain gradient-type models [4]). Geometric objects such as metric tensors, connection coefficients, curvature tensors, and anholonomic objects [5] also depend ultimately only on position. This is true in conventional nonlinear elasticity and plasticity theories [1, 6], as well as geometric theories incorporating torsion and/o curvature tensors associated with crystal defects, for example [7–15]. In these classical theories, the metric tensor is always Riemannian (i.e., essentially dependent only upon or in the spatial or material setting), meaning the length of a differential line element depends only on position; however, torsion, curvature, and/or covariant derivatives of the metric need not always vanish if the material contains various kinds of defects (non-Euclidean geometry). Finsler geometry changed. Another important connection in Riemann-Finsler geometry is the Berwald connections, which was created by Berwald and is neither metrically compatible nor torsionally free. Charles Ehresmann [43] made the first claim that "A manifold is defined by means of an atlas of local charts" in 1943. A connection on bundles called the "Ehresmann Connection" was another invention he made. In 1943, Chern [34], one of the greatest differential geometers of the twentieth century, revealed the important linkages that are almost metrically compatible and torsion-ally free. He addressed the local equivalence and Euclidean connections in Finsler spaces in [35] and reworked it in [36]

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along with the concept of fibre bundles and the theory of connections. As ancient as the calculus of variations, Finsler space is a geometry made up of straightforward integrals of the type (1.2.1). According to Chern [37], "almost all Riemannian geometry results can be constructed in the Finsler framework" in 1996. The next development in the link was the release of Rund's book [133] on Finsler geometry in 1959. He created the "Rund Connection" in this work. Later, the geometrist discovered that it is identical to the Chern connection. For his synthetic solutions to geometrical issues, Busemann, a forerunner of Finsler geometry, gained notoriety. He made an effort to summarise Finsler's argument in [26] [46]. In 1942, he released a book titled "Metric approaches in the Foundations of Geometry and in Finsler Spaces". He emphasises the gaps that safeguard the distinctiveness of geodesics in this body of work. He found a Finsler space that met all of the typical geodesic requirements. Both the geometry of Finsler spaces and the geometry of metric spaces with geodesics are covered. These are referred to as G-spaces, where G is an abbreviation for geodesic with all the usual properties. He released "The Geometry of Geodesics," a revised and extended version of the earlier book, in 1955. In this work, geodesics are shown to exhibit regionally specific properties. In Riemannian geometry as opposed to Finsler geometry, this assumption is more probable. In his paper Rander made the discovery that electromagnetism and the unified field theory of gravity are related. In this investigation, he employed the indicatrix as an eccentric quadratic hypersurface. This includes creating a vector at each location in space and figuring out the indicatrix center's displacement. The length ds formula of a line element must be homogenous to the first degree in dxi. The far more fundamental "eccentric" line component is what this criteria calls for.

$$ds = b_i(x)dx^i + \sqrt{a_{ij}(x)dx^i dx^j}\,,$$

where aij is the basic tensor for Riemannian affine connections and bi is a covariant vector specifying the displacement of the indicatrix centre. The Finsler space was initially used to solve issues in the real world by

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Ingarden. In his work [58], he introduced the Lagrangian formulation for an electron microscope.

2 Design and Methodology

Our study of differential geometry now broadens our knowledge of non-linear features and non-trivial symmetries that occur in several models like classical field theory, quantum field theory, mathematical mechanics, and gravity. Finsler geometry is now used in several ways in the applied sciences. The theory of Yang Mills fields, super-string theory, non linear sigma models, and other categories of non-linear field systems used in contemporary particle theory, quantum gravity, and biology are a few examples of the non-linear field systems that are used in these applications, which started with the traditional field of general relativity. Bao published a book titled "An Introduction to Riemann-Finsler Geometry" [13]. After the release of this book, Riemann-Finsler mathematics and Riemann-Finsler metrics have replaced Finsler geometry and metrics. The fact that the Finsler measure can be applied to psychology is something that every geometer can be quite proud of. Finsler geometry may be used to represent a variety of biological models, including coral reef ecology and protein structure. Rander's metric may handle a number of psychometric problems if non-symmetrical measurements are also permitted. Sometimes a physical purpose comes before a mathematical theory, however this is not always the case. The Riemannian & Finsler geometries were the first pure approaches in differential geometry.

The primary practical application of Riemannian geometry is in general relativity, sometimes referred to as Einstein's "theory of gravity." The use of Finsler geometry and its overall impact on several scientific fields are what motivate its research. The Riemannian approach for designing dynamical systems was first presented by G. Kron [66]. Barthel's stated point Finsler space idea has been extensively used to electron optics by researchers from Japan and Romania in recent years. Theoretical applications of thermodynamics are many, according to Ingarden's essay [59]. The core of Antonelli's Finsler diffusion theory was the tried-and-true Finsler Geometrical method of the Japanese Matsumoto

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school [7]. Modern differential geometry provides a broad variety of instruments for the effective study of Riemannian geometry since it encompasses all results, whether global or local, in an almost direct and refined way and does not have any quadratic limitations. In addition to a realistic and mature understanding of geometry extending from quadric to generic algebraic varieties, this offers a foundational platform for further research. The effort of many geometers throughout the globe has led to the development of this special area of differential geometry, which has significant applications in multiple domains of the natural sciences.

3 Methodlogy and Evaluation

The research on Finsler space having Randers conformally transformed (,)-metric and Finsler space with conformal transformation of second type Douglas space with generalised (,)-metric is presented in the current chapter. Riemannian metrics have a well-developed conformal geometry. Every constant-curvature Riemannian metric is well known to be locally conformally flat. The projective and conformal characteristics of a Finsler space have a unique impact on its metric properties, according to the Weyl theorem in Finsler geometry [65, 133]. As a result, a Finsler metric's conformal characteristics need careful attention. Given that M is an n-dimensional C-manifold, and that F(x, y) and F (x, y) are two Finsler metric functions, let (M, F) be a Finsler space. It is referred to as a conformal change when the change F F, F (x, y) = e (x)F(x,y) occurs when (x) is a function in each coordinate neighbourhood of M. This modification was proposed by Kneblman [65] and investigated by other scholars [53, 60], among others. Conformally flat Finsler metrics are Finsler metrics that are conformally connected to Minkowski metrics. The change is known as a Randers change after Randers, who originally introduced it in [124], where F(x, y) is a Riemannian metric function and = biy i is a 1-form on M. Numerous works have looked at the geometric properties of such a transformation, including [4, 103, 146], etc. The change F (x, y) = F(x, y) + (x, y), where F(x, y) is a Finsler metric function, was introduced by Matsumoto [94] and was given the term -change. He found a relationship between the Cartan

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connection coefficients for (M, F) and (M, F). Numerous writers have since investigated this change, including [103, 145],... etc. All of the aforementioned adjustments have been brought together by Abed in the following manner.

Conformal change is defined as F(x, y) F(x, y) = e(x)F(x, y) + (x, y). where (x, y) = bi(x, y) is a 1-form on M and is a function on x. For instance, the modification (3.1.1) becomes a conformal change if = 0. If F is a Finsler metric or a Riemannian metric function, it shortens to a Randers change when = 0. From the viewpoint of geodesic equations, Bacso and Matsumoto [11] proposed the concept of Douglas space as a generalisation of Berwald space. As a generalisation of Berwald space, we also take the idea of Landsberg space into account.

Weakly-Berwald space is a new generalisation of Berwald space proposed by Bacso and Szilagyi [12]. If the Douglas tensor Dh ijk vanishes identically, then a Finsler space with (,)-metric is said to be a Douglas space of the second sort [98]. Let's define the Douglas type of the second kind. If the homogeneous polynomials Dij = Gi (x, y)y j Gj (x, y)y i are of degree three, then a Finsler space F n is said to be a Douglas space. If and only if Dim m = (n+ 1)Gi Gm my i are homogeneous polynomials of degree two, a Finsler space F n is referred to as a second kind Douglas type. A Finsler space with a (,)-metric, on the other hand, is a Douglas space in [101] if and only if Bij = Bi y j Bj y i are homogeneous polynomials in (y i) of degree three. If and only if Bim m = (n+1)BiBm my i are homogeneous polynomials in (y i) of degree two, a Finsler space of a (,)-metric is said to be a Douglas space of the second type. Different kinds of (,)-metrics have been shown to be conformally invariant in second-kind Douglas space by many authors [82, 107, 136, 126].

Given that Mn is an n-dimensional C-manifold and F(x, y) is a Finsler metric function, let (Mn, F) be a Finsler space. A change is referred to as a conformal change if it occurs when F(x, y) e (x)F(x, y) and if (x) is a function in each coordinate neighbourhood of Mn.

4 Result Analysis

The idea of dual flat Riemannian metrics was first put forth by Amari and Nagaoka [5], who looked at how the major families of probability distributions have different geometrical structures, such as two convex functions connected by Legendre transformations and two flat affine connections that are mutually coupled. Later, Z. Shen [142] studied Finsler information geometry and generalised the dually flatness notations to Finsler metrics without the quadratic limitation. Similar to the Riemannian example, locally dual flat Finsler metrics provide unique geometric characteristics and will be important in Finsler geometry. If there is a coordinate system (x i) at each point with spray coefficients of the following form, then the Finsler metric F = F(x, y) defined on an n-dimensional manifold is said to be locally dual flat.

$$G^i = -\frac{1}{2} g^{ij} H_{y^i},$$

where the scalar function H = H(x, y) is defined on M. According to [142], this kind of coordinate system is an adapted coordinate system. In [33], X Cheng proved that the Finsler metric F = F(x, y) on the open subset U R n is locally-dually flattened if and only if

$$[F^2]_{x^k y^l} y^k - 2[F^2]_{x^l} = 0.$$

It is challenging to directly solve this PDE. The number of locally dual flat metrics that have been found so far is rather small. Unified Riemannian metric

$$F = \sqrt{g_{ij} y^i y^j}$$

$$g_{ij} = \frac{\partial^2 \varphi(x)}{\partial x^i \partial x^j},$$

$$F = \frac{\sqrt{\langle x, y\rangle^2 + (1 - |x|^2)|y|^2} + \langle x, y\rangle}{1 - |x|^2}.$$

When the C function = (x) is present [5]. The Funk-metric, defined on very unit ball B n R n, is the first non-Riemannian locally simultaneously flat metric to be developed [142]. This statistic belongs to the class of unique Randers metrics. The author looked at locally dual flat Randers measures in [33]. In 2011, Xia [163] provided the comparable requirements of locally dual flat (,)-metrics on a manifold with dimension n 3, and he also found the locally dual flat Finsler metric with isotropic flag curvature [162]. Yang Li [89] examined the corresponding circumstances for a square metric for a locally dual flat (,)-metric in 2019. The study of the Finsler metric using reversible geodesics is an intriguing idea in Finsler geometry. The research of Finsler metrics using reversible geodesics is a distinctive area of study in Finsler geometry, as well as in physics. A Finsler space is said to have reversible geodesics if every oriented geodesic route is also a geodesic when it is taken in the opposite direction. The study of reversible geodesics in Finsler space is addressed in [40], Crampin investigates reversible geodesics in Randers space, Masca discusses reversible geodesics in [92], and Sabau discusses reversible geodesics in Finsler manifolds on [134]. We developed the idea of Finsler space of reversible geodesics with a generalised (,)-metric in light of the aforementioned research publications.

5 Conclusion

Under certain circumstances, we were able to achieve the findings for the locally dually flat generalised (,)-metric F = 1 + 2 + 3 2. With the use of the generalised first approximate Matsumoto metric and the generalised (,)-metric, we were able to investigate if there are any reversible geodesics in Finsler space. The requirements for Finsler space F on M to be reversible geodesics have been established. Additionally, we have used reversible geodesics to study different aspects of F's geometry. A weighted quasi metric dF on M is produced by the generalised (,)-metric F, we have shown. Finally, using a generalised (,)-metric, we were able to achieve the T-tensor result.

In this chapter, we looked at the Chua circuit system's cubic nonlinear function's Jacobi stability characteristics. We reconstructed the modified Chua circuit system as a pair of second-order nonlinear differential equations in order to use KCC theory. Then, we discovered the Berwald connection, five geometrical (KCC) invariants, and the nonlinear connection. Except for the second KCC invariant, all KCC invariants disappear. The deviation tensor (second invariants) is produced, and its irregularly behaving components' temporal variations are shown.

Additionally, we have shown how the determinant, eigenvalues, and trace change over time. We first determine the deviation tensor components at each equilibrium point before determining the Jacobi stability requirements.

Near each equilibrium point, the deviation vector's temporal fluctuation and the instability exponents are shown. We have also shown the deviation vector's temporal fluctuation in curvature, which illustrates the system's chaotic character.

We take into account the geometric setup mentioned in Section 3.1 for a 2-dimensional spray S. As we just noted, formula (3.1.5) determines S's Jacobi endomorphism if S is isotropic. The semi-basic 1-form and the Ricci scalar are not everywhere vanishing on T0M since we assume that S is not flat. We shall limit the domain to some open cone A T0M, where and are not vanishing, if we are able to deal with conic Finsler functions.

H is a horizontal vector field that is 2+-homogeneous based on the first two requirements of (3.4.1). The last condition above corresponds to (H) = JH, which denotes that H is (fibrewise) an eigenvector for the Jacobi endomorphism and represents the non-vanishing eigenvalue.

6 Reference

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